"HCR's or H. Rajpoot's Formula for Regular Polyhedron"

$$\textit{Mathematical Analysis of Regular Polyhedron}$$

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1. Introduction: We very well know that a regular n-polyhedron is the solid which has all its faces as congruent regular n-polygons. Each face as a regular n-polygon of any **regular polyhedron** (out of five platonic solids **having convex surface)** corresponds to the base of a right pyramid with apex point at the centre. Thus, it can be divided into a number of congruent elementary right pyramids equal to the number of faces. Here, we are to calculate the important parameters as inner radius (R_i), outer radius (R_o), mean radius (R_m), surface area (A_s) & volume (V) of a given regular polyhedron just by co-relating the different parameters of elementary right pyramids with that of a regular polyhedron. **A regular polyhedron always has convex surface i.e. any of five platonic solids (i.e. regular tetrahedron, cube, regular octahedron, regular icosahedron, regular dodecahedron)**

2. Elementary right pyramid: A right pyramid, having base as a regular n-polygon same as the face of a given regular polyhedron & its apex point at the centre of that polyhedron, is called elementary right pyramid for that regular polyhedron. Elementary right pyramids of a regular polyhedron are always congruent. Hence, by joining all these congruent elementary right pyramids of a regular polyhedron, keeping their apex points coincident with each other, the original (parent) polyhedron is obtained or constructed.

For ease of understanding & calculations for a given regular polyhedron, let's consider an elementary right pyramid having base as a regular n-polygon. Hence, by calculating the vertical height (H), length of lateral edge (CA_1), area (A_b) of regular n-polygonal base (or face), volume (V_1) & angle (α) between lateral edges of an elementary right pyramid we can easily calculate all the important parameters as inner radius (R_i), outer radius (R_o), mean radius (R_m), surface area (A_s) & volume (V) of a given regular polyhedron. For calculation, let's take the following assumptions/parameters

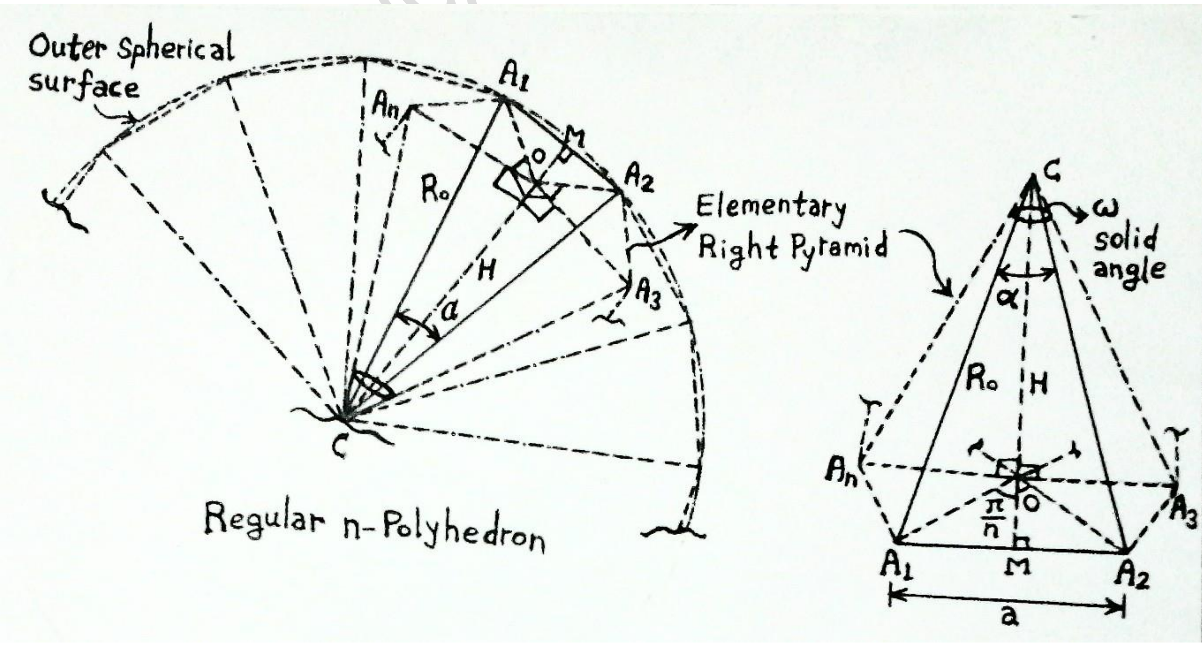

Figure 1: An elementary right pyramid with base as a regular n-polygon is taken out from a regular polyhedron

3. Assumptions/Parameters: Let's us assume the following parameters for a given **regular polyhedron having all its faces as the congruent regular n-polygons**

$\boldsymbol{n_f} = no. of\ faces\ of\ regular\ polyhedron = no. of\ elementary\ right\ pyramids \qquad \forall\ n_f \in N\ \&\ n_f \geq 4$

$\boldsymbol{n} = no. of\ sides\ in\ each\ face\ (i.e.\ no.\ of\ sides\ in\ each\ regular\ polygon) \qquad \forall\ n \in N\ \&\ n \geq 3$

$= no. of\ edges\ in\ each\ face\ (n_e)$

$\boldsymbol{\alpha} = angle\ between\ any\ two\ consecutive\ lateral\ edges\ in\ the\ elementary\ right\ pyramid$

$= \boldsymbol{Edge\ angle\ (parametric\ angle)\ of\ elementary\ right\ pyramid} \qquad (\forall\ \alpha = f(n, n_f)\ \&\ n < n_f)$

$\boldsymbol{a} = edge\ length\ i.e. length\ of\ side\ of\ each\ face$

$H = normal\ height\ of\ the\ elementary\ right\ pyramid$

$\omega = solid\ angle\ subtended\ by\ each\ of\ the\ faces\ at\ the\ centre\ of\ polyhedron$

$= solid\ angle\ subtended\ by\ elementary\ right\ pyramid\ at\ its\ apex\ point\ (i.e.\ centre\ of\ polyhedron)$

$R_i = inner\ radius\ (radius\ of\ an\ imaginary\ spherical\ surface\ touching\ all\ the\ faces\ of\ polyhedron)$

$= H\ (vertical\ height\ of\ elementary\ right\ pyramid)$

$R_o = outer\ radius\ i.e.\ radius\ of\ an\ imaginary\ spherical\ surface\ pasing\ through\ all\ the\ vertices$

$= length\ of\ lateral\ edge\ of\ elementary\ right\ pyramid\ (CA_1)$

R_m
$= mean\ radius\ (radius\ of\ an\ imaginary\ spherical\ surface\ having\ volume\ equal\ to\ that\ of\ polyhedron)$

$A_s = surface\ area\ of\ polyhedron = no. of\ faces \times area\ of\ face\ (base\ of\ ele.\ pyramid) = n_f \times A_b$

$V = volume\ of\ polyhedron = no. of\ faces \times volume\ of\ elementary\ right\ pyramid = n_f \times V_1$

Among all above parameters, angle α is dependent (as a function) only on the two mutually dependent positive integers $n\ \&\ n_f$. Thus, **angle $\boldsymbol{\alpha}$ is the deciding parameter for construction of a regular polyhedron**.

Let's consider an elementary right pyramid with base as a regular polygon $A_1A_2A_3 \dots . A_n$ having 'n' no. of sides each of length 'a', angle between any two consecutive lateral edges 'α' & normal height 'H' taken out from a given regular polyhedron (as shown in the figure 1 above)

Now, join all the vertices $A_1, A_2, A_3, \dots . A_n$ of base to the centre 'O' thus, we obtain 'n' no. of congruent isosceles triangles $\Delta A_1OA_2, \Delta A_2OA_3 \dots \dots . \dots . \Delta A_nOA_1$

Now, in right ΔOMA_1

$$\Rightarrow \tan \alpha A_1OM = \frac{MA_1}{OM}$$

$$or\ \ tan\frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{OM} \qquad \Rightarrow\ OM = \frac{a}{2}cot\frac{\pi}{n} \quad \left(since, \alpha A_1OA_2 = \frac{2\pi}{n}\right)$$

In rightΔCOM

$$\Rightarrow CM^2 = OC^2 + OM^2 \;\; or \; CM = \sqrt{H^2 + \left(\frac{a}{2}cot\frac{\pi}{n}\right)^2}$$

$$\Rightarrow \boldsymbol{CM = \frac{1}{2}\sqrt{4H^2 + a^2cot^2\frac{\pi}{n}}} \qquad \dots\dots\dots\dots\dots..(I)$$

In right ΔCMA_2

$$\Rightarrow \tan \alpha A_2CM = \frac{MA_2}{CM}$$

$$\Rightarrow tan\frac{\alpha}{2} = \frac{\left(\frac{a}{2}\right)}{CM} \;\; or \;\; \boldsymbol{CM = \frac{a}{2}cot\frac{\alpha}{2}} \qquad \dots\dots\dots\dots\dots\dots.(II)$$

Now, equating the values of **CM** from equation (I) & (II), we have

$$\Rightarrow \frac{a}{2}cot\frac{\alpha}{2} = \frac{1}{2}\sqrt{4H^2 + a^2cot^2\frac{\pi}{n}}$$

On squaring both the sides, we get

$$a^2cot^2\frac{\alpha}{2} = 4H^2 + a^2cot^2\frac{\pi}{n} \qquad \Rightarrow 4H^2 = a^2\left(cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}\right)$$

$$H = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}$$

4. Inner (inscribed) radius (R_i) of regular polyhedron: It is the radius of an imaginary spherical surface touching all the faces of a regular polyhedron, it's always equal to the normal height (H) of the elementary right pyramid thus we have

$$\boldsymbol{R_i = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}} \qquad \dots\dots\dots\dots\dots\dots\dots\dots.(III)$$

5. Outer (circumscribed) radius (R_o) of regular polyhedron: It is the radius of an imaginary spherical surface passing through all the vertices of a regular polyhedron. It's is equal to the length of lateral edge of elementary right pyramid & is calculated as follows

In right ΔCMA_1

$$\Rightarrow \sin \alpha A_1CM = \frac{MA_1}{CA_1}$$

$$or \;\; sin\frac{\alpha}{2} = \frac{\left(\frac{a}{2}\right)}{CA_1} \qquad \Rightarrow CA_1 = \frac{a}{2}cosec\frac{\alpha}{2} \;\; (since, \alpha A_1CA_2 = \alpha)$$

$$\boldsymbol{R_o = \frac{a}{2}cosec\frac{\alpha}{2}} \qquad \dots\dots\dots\dots\dots\dots\dots.(IV)$$

5. Surface area (A_s) of regular polyhedron: We know that the base of right pyramid is a regular n-polygon hence the surface area of regular polyhedron is the sum area of all the faces & is calculated as follows

Area of isosceles ΔA_1OA_2 (See above figure 1)

$$=\frac{1}{2}\times(A_1A_2)(OM)=\frac{1}{2}\times(a)\left(\frac{a}{2}\cot\frac{\pi}{n}\right)=\frac{1}{4}a^2\cot\frac{\pi}{n}$$

Since, the base of elementary right pyramid consists of ‘n’ number of isosceles triangles congruent to ΔA_1OA_2 hence the area of the base (i.e. face of regular polyhedron), A_b

$$\Rightarrow \boldsymbol{A_b}=n\times(area\ of\ \Delta A_1OA_2\ )=n\times\frac{1}{4}a^2\cot\frac{\pi}{n}=\boldsymbol{\frac{1}{4}na^2\cot\frac{\pi}{n}}$$

Since, the regular n-polyhedron has n_f no. of congruent faces each as a regular n-polygon hence the surface area A_s of regular n-polyhedron

$$\boldsymbol{A_s}=n_f\times(area\ of\ one\ face,\ \ A_b)=n_f\times\left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)=\frac{1}{4}nn_fa^2\cot\frac{\pi}{n}$$

$$\Rightarrow \boldsymbol{A_s=\frac{1}{4}nn_fa^2\cot\frac{\pi}{n}} \qquad \dots\dots\dots\dots\dots\dots\dots(V)$$

6. Volume (V) of regular polyhedron: Volume (V_1) of the elementary right pyramid with base as a regular n-polygon is calculated by using **analogy of right pyramid with a right cone** as follows

$$\boldsymbol{V_1}=\frac{1}{3}(area\ of\ base)(normal\ height)=\frac{1}{3}(A_b)(H)$$

$$=\frac{1}{3}\left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)\left(\frac{a}{2}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}\right) \qquad (by\ setting\ the\ value\ of\ H\ from\ eq(III))$$

$$=\boldsymbol{\frac{1}{24}na^3\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}}$$

Since, the regular polyhedron has n_f no. of congruent elementary right pyramids hence the volume (V) of regular polyhedron is the sum of all these elementary volumes & is given as follows

$$\boldsymbol{V}=n_f\times(volume\ of\ elementary\ right\ pyramid,\ \ V_1)$$

$$=n_f\times\left(\frac{1}{24}na^3\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}\right)=\frac{1}{24}nn_fa^3\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}$$

$$\Rightarrow \boldsymbol{V=\frac{1}{24}nn_fa^3\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}} \qquad \dots\dots\dots\dots\dots\dots\dots(VI)$$

7. Mean radius (R_m) of regular polyhedron: It is the radius of an imaginary spherical surface whose volume is equal to the volume of a given regular polyhedron. Hence it is calculated by equating volume of spherical surface to that of regular polyhedron as follows

$$\frac{4\pi}{3}{R_m}^3=\frac{1}{24}nn_fa^3\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}$$

$$\Rightarrow {R_m}^3=\frac{1}{32\pi}nn_fa^3\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}$$

$$R_m=\left(\frac{1}{32\pi}nn_fa^3\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}\right)^{\frac{1}{3}}$$

$$\Rightarrow \boldsymbol{R_m = a\left(\frac{nn_f}{32\pi}\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}\right)^{\frac{1}{3}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots.(VII)$$

$\boldsymbol{For\ finite\ value\ of\ no.of\ faces\ (n_f)\ \Rightarrow R_i < R_m < R_o}$

It implies that a regular polyhedron is equivalent to a spherical surface (or sphere) having a radius equal to the mean radius (R_m) of that regular polyhedron for the same volume.

8. Need to apply "HCR's Theory of Polygon" to derive a formula (HCR's Formula): It is clear from the eq(III), (IV), (VI) & (VII) that the **inner radius, outer radius, volume & mean radius** of a regular polyhedron depends on **a particular parametric angle α** i.e. **edge angle of elementary right pyramid**. Hence, in order to determine the value of unknown edge angle $\boldsymbol{\alpha}$**,** we have to take out an elementary right pyramid from a given regular polyhedron which is practically difficult to deconstruct the polyhedron & measure the unknown angle (α) with the desired accuracy. **Hence, if there is an error in the value of unknown angle α then there would be error in the values of inner radius, outer radius, volume & mean radius of a regular polyhedron.** But this difficulty is very easily overcome by deriving a formula just named as **HCR's Formula for a regular polyhedron** to calculate this particular unknown parametric angle $\boldsymbol{\alpha}$ **by applying HCR's Theory of Polygon. This formula doesn't depend on any of the geometrical dimensions of a regular polyhedron but it merely depends on the no. of the faces (n_f) & no. of the edges in one face (n) hence it's also called as HCR's dimensionless formula.**

9. Derivation of HCR's Formula for regular polyhedron: Value of deciding parametric angle α for any regular polyhedron:

We know from **HCR's theory of polygon,** solid angle (ω), subtended by a regular polygon (i.e. plane bounded by straight lines) having n no. of sides each of length a at any point lying at a normal distance (height) H from the centre of plane, is given as

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}}\right)$$

(Note: Above result is directly taken without any proof as it has already derived in "HCR's Theory of Polygon")

Since each of the congruent faces, as a regular n-polygon, of regular polyhedron is at a normal distance H from the centre, hence by setting the value of H from the eq(III) in above formula, we get the value of **solid angle (ω) subtended by each regular n-polygonal face at the centre of given regular polyhedron**

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2\left(\frac{a}{2}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}\right)\sin\frac{\pi}{n}}{\sqrt{4\left(\frac{a}{2}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}\right)^2 + a^2\cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{\sin\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}}{\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}}}\right) = 2\pi - 2n\sin^{-1}\left(\frac{\sin\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}}{\cot\frac{\alpha}{2}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{\sin\frac{\pi}{n}\sqrt{\frac{1}{\tan^2\frac{\alpha}{2}} - \frac{1}{\tan^2\frac{\pi}{n}}}}{\cot\frac{\alpha}{2}}\right) = 2\pi - 2n\sin^{-1}\left(\frac{\sin\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}}{\cot\frac{\alpha}{2}\tan\frac{\alpha}{2}\tan\frac{\pi}{n}}\right)$$

$$\Rightarrow \boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right)} \quad \dots\dots\dots\dots\dots..(VIII)$$

Since, there are total n_f no. of faces as congruent regular n-polygons in a given regular polyhedron hence each of the faces subtends an equal solid angle (ω) at the centre of polyhedron

$$\therefore\ solid\ angle\ (\omega)\ subtended\ by\ each\ of\ n_f\ no.of\ \ faces\ at\ the\ centre\ of\ regular\ polyhedron$$

$$= \frac{total\ solid\ angle}{n_f} = \frac{4\pi}{n_f}$$

$$\Rightarrow\ \boldsymbol{\omega = \frac{4\pi}{n_f}} \quad \dots\dots\dots\dots\dots\dots\dots.(IX)$$

Now, **equating both the results from eq(VIII) & eq(IX), we have**

$$2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right) = \frac{4\pi}{n_f}$$

$$\Rightarrow\ 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right) = 2\pi\left(1 - \frac{2}{n_f}\right)$$

$$\Rightarrow\ \cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}} = \sin\left\{\frac{\pi}{n}\left(1 - \frac{2}{n_f}\right)\right\}$$

$$or\ \ \sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}} = \frac{\sin\left\{\frac{\pi}{n}\left(1 - \frac{2}{n_f}\right)\right\}}{\cos\frac{\pi}{n}}$$

$$\Rightarrow\ \tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2} = \frac{\sin^2\left\{\frac{\pi}{n}\left(1 - \frac{2}{n_f}\right)\right\}}{\cos^2\frac{\pi}{n}} \quad (squaring\ both\ sides)$$

$$\Rightarrow\ \tan^2\frac{\alpha}{2} = \tan^2\frac{\pi}{n} - \frac{\sin^2\left\{\frac{\pi}{n}\left(1 - \frac{2}{n_f}\right)\right\}}{\cos^2\frac{\pi}{n}} = \frac{\sin^2\frac{\pi}{n}}{\cos^2\frac{\pi}{n}} - \frac{\sin^2\left\{\frac{\pi}{n}\left(1 - \frac{2}{n_f}\right)\right\}}{\cos^2\frac{\pi}{n}}$$

$$or\ \ \tan^2\frac{\alpha}{2} = \frac{\sin^2\frac{\pi}{n} - \sin^2\left\{\frac{\pi}{n}\left(1 - \frac{2}{n_f}\right)\right\}}{\cos^2\frac{\pi}{n}}$$

$$\Rightarrow\ tan\frac{\alpha}{2}=\sqrt{\frac{sin^2\frac{\pi}{n}-sin^2\left\{\frac{\pi}{n}\left(1-\frac{2}{n_f}\right)\right\}}{cos^2\frac{\pi}{n}}}=\frac{\sqrt{sin^2\frac{\pi}{n}-sin^2\left\{\frac{\pi}{n}\left(1-\frac{2}{n_f}\right)\right\}}}{cos\frac{\pi}{n}}$$

$$\Rightarrow\ tan\frac{\alpha}{2}=sec\frac{\pi}{n}\sqrt{sin\left\{\frac{\pi}{n}-\frac{\pi}{n}\left(1-\frac{2}{n_f}\right)\right\}sin\left\{\frac{\pi}{n}+\frac{\pi}{n}\left(1-\frac{2}{n_f}\right)\right\}}$$

$$since,\qquad sin^2A-sin^2B=\sin(A-B)sin(A+B)$$

$$\therefore\ tan\frac{\alpha}{2}=sec\frac{\pi}{n}\sqrt{sin\left\{\frac{2\pi}{nn_f}\right\}sin\left\{\frac{2\pi}{n}\left(1-\frac{1}{n_f}\right)\right\}}=sec\frac{\pi}{n}\sqrt{sin\left\{\frac{2\pi}{nn_f}\right\}sin\left\{\frac{2\pi}{n}\left(\frac{n_f-1}{n_f}\right)\right\}}$$

$$\Rightarrow\ \left[\boldsymbol{\alpha=2\tan^{-1}\left(sec\frac{\pi}{n}\sqrt{sin\left\{\frac{2\pi}{nn_f}\right\}sin\left\{\frac{2\pi(n_f-1)}{nn_f}\right\}}\right)}\right]\qquad\forall\ \boldsymbol{\alpha\leq\frac{2\pi}{n}}$$

$$\textbf{Where},\ \boldsymbol{n,n_f\in N,\ \ n\geq 3,\ \ n_f\geq 4\ \&\ n<n_f}$$

$$\boldsymbol{or\quad \alpha=f(n,n_f)\qquad \forall\ n,n_f\in N\ \Rightarrow\ n\geq 3,\ \ n_f\geq 4\ \ \&\ \ n<n_f}$$

Above formula is in the generalised form called ***HCR's Formula for Regular Polyhedron*** it is applied on any **regular polyhedron** (**any of five existing platonic solids**) to calculate the value of unknown parametric angle α. In this formula, there are only two arbitrary constants $n\ \&\ n_f$ which are always positive integers & vary dependently. **Edge angle $\boldsymbol{\alpha}$ of the elementary right pyramid of an existing/constructed regular polyhedron is the function of two variables $\boldsymbol{n\ \&\ n_f}$ only i.e. edge angle $\boldsymbol{\alpha}$ doesn't depend on any of the geometrical dimensions of a regular polyhedron.**

Thus, we find that the value of unknown edge angle $\boldsymbol{\alpha}$ of elementary right pyramid of any given regular polyhedron is determined simply by counting the no. of faces ($\boldsymbol{n_f}$) & the no. of edges ($\boldsymbol{n}$) in one face only.

Note: For more ease of understanding & familiarity of above formula, let the no. of edges in one face be denoted/symbolised by $\boldsymbol{n_e}$ then HCR's Formula will take a look as follows

$$\left[\boldsymbol{\alpha=2\tan^{-1}\left(sec\frac{\pi}{n_e}\sqrt{sin\left\{\frac{2\pi}{n_en_f}\right\}sin\left\{\frac{2\pi(n_f-1)}{n_en_f}\right\}}\right)}\right]\ \ \forall\ \ \boldsymbol{\alpha\leq\frac{2\pi}{n_e}}$$

$$\boldsymbol{where,\ \ n_e,n_f\in N,\ \ n_e\geq 3,\ \ n_f\geq 4\ \&\ n_e<n_f}$$

$$\boldsymbol{or\quad \alpha=f(n_e,n_f)\qquad \forall\ n_e,n_f\in N\ \Rightarrow\ n_e\geq 3,\ \ n_f\geq 4\ \ \&\ \ n_e<n_f}$$

Notation (n_e) is used only for the sake of better understanding of the symbols used in the formula. All other things being the same, but here, we would use first original form only. For desired changes, we have

$$n_f=no.of\ faces\ as\ congruent\ regular\ polygons\qquad\qquad \forall\ n_f\geq 4$$

$$n_e=no.of\ edges\ or\ sides\ in\ one\ face\ of\ Regular\ Polyhedron\quad \forall\ n_e\ \geq 3$$

Remember: n_e & n_f are mutually dependent positive integers & having particular relation between them for an existing regular polyhedron because the existence or non-existence of a regular polyhedron depends on these positive integers only i.e. it doesn't depend on any of the geometrical dimensions of regular polyhedron.

10. Solution of an existing regular polyhedron: In order to find out the solution of all the parameters as **inner radius, outer radius, mean radius, surface area & volume of any of five regular polyhedrons i.e. platonic solids,** three important values of n, n_f & a should be known because n & n_f are always positive integers & the no. of edges in one face (n) & the no. of faces (n_f) can never be fractions & also can never be obtained as the fractions in any reverse calculations relevant to any existing regular polyhedron. Thus, we are bounded to these positive integer values with their minimum limits for an existing regular polyhedron. For example, if a regular polyhedron has all its faces as congruent pentagons $(n = 5)$ then it will exist only & if only it has total twelve no. of congruent regular pentagonal faces $(n_f = 12)$ otherwise it would not exist. Hence n & n_f are mutually dependent integers. That's why there are only five existing regular polyhedrons i.e. platonic solids.

Let's follow the following steps to calculate all the important parameters of any existing regular polyhedron

Step 1: First of all, calculate the value of edge angle (α) of elementary right pyramid simply by setting the known values of $n\ (no. of\ edges\ in\ one\ face)$& $n_f\ (no. of\ faces)$ of a given regular polyhedron in HCR's formula as follows

$$\boldsymbol{\alpha = 2\tan^{-1}\left(sec\frac{\pi}{n}\sqrt{sin\left\{\frac{2\pi}{nn_f}\right\}sin\left\{\frac{2\pi(n_f-1)}{nn_f}\right\}}\right)\quad \forall\, \alpha \le \frac{2\pi}{n}}$$

Step 2: Set this known value of edge angle (α) of elementary right pyramid in all the necessary formula to calculate desired parameters of regular polyhedron as tabulated below

Inner Radius $(R_i) \rightarrow$	$\boldsymbol{R_i = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}}$
Outer Radius $(R_o) \rightarrow$	$R_o = \frac{a}{2}cosec\frac{\alpha}{2}$
Mean Radius $(R_m) \rightarrow$	$R_m = a\left(\frac{nn_f}{32\pi}cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}\right)^{\frac{1}{3}}$ ***For finite value of no. of faces*** $(n_f) \Rightarrow \boldsymbol{R_i < R_m < R_o}$
Surface Area $(A_s) \rightarrow$	$A_s = \frac{1}{4}nn_fa^2cot\frac{\pi}{n}$
Volume $(V) \rightarrow$	$V = \frac{1}{24}nn_fa^3cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}$

11. Construction of regular n-polyhedron (platonic solid):

Construction of an existing regular polyhedron is very simple if we construct its all the congruent elementary right pyramids with their known parameters. Before construction of any regular polyhedron, a designer has to pre-set the no. of sides (n) in one face & the no. of faces (n_f) such that all n_f no. of regular n-polygonal faces must be exactly interlocked edge-wise to form a closed regular polyhedron (i.e. platonic solid). Although, edge length (a) can be modified according to the requirements of inner radius, outer radius, mean radius, surface area or volume of a regular polyhedron Let's follow the following steps to construct an existing regular polyhedron (There are total five regular polyhedrons i.e. platonic solids)

Step 1: First of all, construct the base (i.e. face of polyhedron) of elementary right pyramid as a regular polygon with n no. of sides of equal length a $(i.e. edge\ length\ of\ face\ of\ polyhedron)$ & each side subtends a plane angle $\left(2\pi/_n\right)$ radian at the centre of base as a regular n-polygon.

Step 2: Now, construct each elementary right pyramid with known base & its vertical height H as follows

$$H = inner\ radius\ of\ regular\ polyhedron = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}$$

& each elementary right pyramid has a volume (V_1)

$$V_1 = \frac{1}{24}na^3 cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}$$

We can fill up the above volume (V_1) with desirable material to construct a solid regular polyhedron.

Step 3: Now construct total n_f no. of congruent elementary right pyramids with the help of above known dimensions of regular n-polygonal base & normal height. Paste/bond by overlapping the lateral faces & interlocking regular n-polygonal bases of all these n_f no. of congruent elementary right pyramids keeping their apex points coincident with each other to construct the required regular polyhedron. Thus, we can construct any regular n-polyhedron (out of total five platonic solids) with the desired dimension i.e. edge length. Thus , any of five platonic solids can be easily constructed with desired dimension (edge length).

12. Important deductions from HCR's Dimensionless Formula:

1. Edge angle (α) is a function of arbitrary positive integers n & n_f i.e. $\alpha = f(n, n_f)$ from HCR's Formula hence **for all the existing regular polyhedrons (five platonic solids) in the universe having equal no. of congruent regular n-polygonal faces (n_f), the edge angle (α) of their elementary right pyramids is same irrespective of their geometrical dimensions like inner radius, outer radius, mean radius, surface area or volume**.

2. If the no. of faces (n_f) & the no. of edges in one face (n) in a given regular n-polyhedron are kept constant then the value of edge angle α is not changed at all because it is a function of only two arbitrary positive integers n & n_f i.e. $\alpha = f(n, n_f)$ from HCR's Formula. Now the variations in different parameters **inner radius, outer radius, mean radius, surface area & volume** of the regular n-polyhedron can be determined by changing the value of edge length (a) as follows

a. Variation of inner (inscribed) radius (R_i) of regular polyhedron

$$R_i = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}} \Rightarrow \boldsymbol{R_i \propto a\,(edge\ length)}$$

$$\Rightarrow inner\ radius\ (R_i)\ is\ directly\ proportional\ to\ the\ edge\ length\ (a)\ in\ a\ regular\ polyhedron$$

b. Variation of outer (circumscribed) radius (R_o) of regular polyhedron

$$R_o = \frac{a}{2}cosec\frac{\alpha}{2} \Rightarrow \boldsymbol{R_o \propto a\,(edge\ length)}$$

$$\Rightarrow outer\ radius\ (R_o)\ is\ directly\ proportional\ to\ the\ edge\ length\ (a)\ in\ a\ regular\ polyhedron$$

c. Variation of mean radius (R_m) of regular polyhedron

$$R_m = a\left(\frac{nn_f}{32\pi}cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}\right)^{\frac{1}{3}} \Rightarrow \boldsymbol{R_m \propto a\,(edge\ length)}$$

$$\Rightarrow mean\ radius\ (R_m)\ is\ directly\ proportional\ to\ the\ edge\ length\ (a)\ in\ a\ regular\ polyhedron$$

d. Variation of surface area (A_s) of regular polyhedron

$$A_s = \frac{1}{4}nn_f a^2 cot\frac{\pi}{n} \ \Rightarrow\ \boldsymbol{A_s \propto a^2(square\ of\ edge\ length)}$$

$$\Rightarrow surface\ area\ (A_s)\ is\ directly\ proportional\ to\ the\ square\ of\ edge\ length\ (a)\ in\ a\ regular\ polyhedron$$

e. Variation of volume (V) of regular polyhedron

$$V = \frac{1}{24}nn_f a^3 cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}} \ \Rightarrow\ \boldsymbol{V \propto a^3(cube\ of\ edge\ length)}$$

$$\Rightarrow volume\ (V)\ is\ directly\ proportional\ to\ the\ cube\ of\ edge\ length\ (a)\ in\ a\ regular\ polyhedron$$

Example: A designer is to design an existing regular polyhedron having n_f no. of congruent regular n-polygonal coloured faces but for the clarity of faces of polyhedron he just increased the edge length of all the faces twice the original value then the inner, outer & mean radii of polyhedron would be twice the original value. While the surface area becomes $2^2 = 4$ times & the volume of polyhedron becomes $2^3 = 8$ times the original value

Analysis of Platonic Solids

There are total five existing regular polyhedrons called platonic solids which have all the faces as congruent regular n-polygons. Let's calculate the important parameters of all the five regular polyhedrons (platonic solids) as given below by applying formula

1. $n = 3$ & $n_f = 4$: Regular tetrahedron having four congruent faces each as an equilateral triangle.
2. $n = 4$ & $n_f = 6$: Regular hexahedron i.e. cube having six congruent faces each as a square.
3. $n = 3$ & $n_f = 8$: Regular octahedron having eight congruent faces each as an equilateral triangle.
4. $n = 5$ & $n_f = 12$: Regular dodecahedron having twelve congruent faces each as a regular pentagon
5. $n = 3$ & $n_f = 20$: Regular icosahedron having twenty congruent faces each as an equilateral triangle

Analysis of Regular Tetrahedron

We know that a **regular tetrahedron** is the simplest & the smallest regular polyhedron (out of five platonic solids) having **four congruent faces each as an equilateral triangle**. It can be easily constructed by joining four congruent right pyramids with base as an equilateral triangle & edge angle (α) (i.e. angle between lateral edges of elementary right pyramid). Now, let there be a regular tetrahedron having each edge of length a. We can easily find out all the parameters of a regular tetrahedron by using HCR's formula to calculate edge angle α as follows

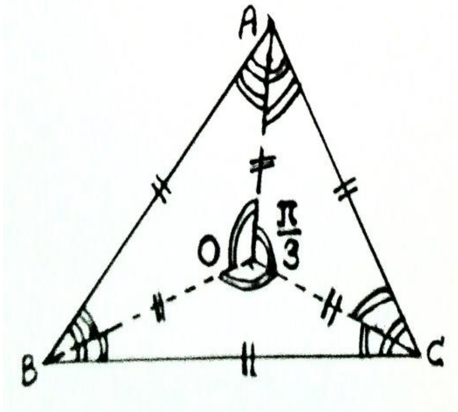

Figure 2: Regular Tetrahedron having four congruent equilateral triangular faces $(\boldsymbol{n=3}$ & $\boldsymbol{n_f=4})$

$$\alpha = 2\tan^{-1}\left(sec\frac{\pi}{n}\sqrt{sin\left\{\frac{2\pi}{nn_f}\right\}sin\left\{\frac{2\pi(n_f-1)}{nn_f}\right\}}\right) \quad \forall\, \alpha \le \frac{2\pi}{n}$$

In this case of a regular tetrahedron, we have

$$n = no.of\ edges\ in\ one\ face = 3\ \ \&\ n_f = no.of\ faces = 4$$

Now, setting both these integer values in HCR's Formula, we get

$$\alpha = 2\tan^{-1}\left(sec\frac{\pi}{3}\sqrt{sin\left\{\frac{2\pi}{3\times 4}\right\}sin\left\{\frac{2\pi(4-1)}{3\times 4}\right\}}\right) = 2\tan^{-1}\left(sec\frac{\pi}{3}\sqrt{sin\left\{\frac{\pi}{6}\right\}sin\left\{\frac{\pi}{2}\right\}}\right)$$

$$= 2\tan^{-1}\left(2\sqrt{\frac{1}{2}\times 1}\right) = \mathbf{2\tan^{-1}(\sqrt{2}) = 2\cot^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 109.4712206^o}$$

The above value of edge angle (α) shows that the angle between any two bonds in a tetrahedral structure of molecule ex. methane (CH_4)

Now, by setting $\alpha = 2\cot^{-1}\left(1/\sqrt{2}\right)$, $n = 3,\ n_f = 4$ & $edge\ length = a$, all the parameters of a regular tetrahedron are calculated as follows

1. Inner (inscribed) radius (R_i)

$$\boldsymbol{R_i} = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}} = \frac{a}{2}\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - cot^2\frac{\pi}{3}} = \frac{a}{2}\sqrt{\frac{1}{2}-\frac{1}{3}} = \frac{a}{2}\sqrt{\frac{1}{6}}$$

$$= \boldsymbol{\frac{a}{2\sqrt{6}} \approx 0.204124145a}$$

Hence, the **volume ($V_{inscribed}$) of the largest sphere inscribed/trapped in a regular tetrahedron**

$$\Rightarrow \boldsymbol{V_{inscribed}} = \frac{4}{3}\pi(R_i)^3 = \frac{4}{3}\pi\left(\frac{a}{2\sqrt{6}}\right)^3 = \boldsymbol{\frac{\pi a^3}{36\sqrt{6}} \approx 0.035626384a^3}$$

2. Outer (circumscribed) radius (R_o)

$$\boldsymbol{R_o} = \frac{a}{2}cosec\frac{\alpha}{2} = \frac{a}{2}\sqrt{1+\left(cot\frac{\alpha}{2}\right)^2} = \frac{a}{2}\sqrt{1+\left(\frac{1}{\sqrt{2}}\right)^2} = \boldsymbol{\frac{a}{2}\sqrt{\frac{3}{2}} \approx 0.612372435a}$$

Hence, the **volume ($V_{circumscribed}$) of the smallest sphere circumscribing a regular tetrahedron**

$$\Rightarrow \boldsymbol{V_{circumscribed}} = \frac{4}{3}\pi(R_o)^3 = \frac{4}{3}\pi\left(\frac{a}{2}\sqrt{\frac{3}{2}}\right)^3 = \frac{\boldsymbol{\pi a^3}}{\boldsymbol{4}}\sqrt{\frac{\boldsymbol{3}}{\boldsymbol{2}}} \approx \boldsymbol{0.961912372a^3}$$

3. Mean radius (R_m)

$$R_m = a\left(\frac{nn_f}{32\pi}cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}\right)^{\frac{1}{3}} = a\left(\frac{3\times 4}{32\pi}cot\frac{\pi}{3}\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - cot^2\frac{\pi}{3}}\right)^{\frac{1}{3}}$$

$$\boldsymbol{R_m} = a\left(\frac{3}{8\pi}\left(\frac{1}{\sqrt{3}}\right)\sqrt{\frac{1}{2}-\frac{1}{3}}\right)^{\frac{1}{3}} = a\left(\frac{\sqrt{3}}{8\pi}\sqrt{\frac{1}{6}}\right)^{\frac{1}{3}} = a\left(\frac{\sqrt{3}}{8\pi}\sqrt{\frac{1}{6}}\right)^{\frac{1}{3}} = \frac{\boldsymbol{a}}{\left(\boldsymbol{8\pi\sqrt{2}}\right)^{1/3}} \approx \boldsymbol{0.304145723a}$$

$$from\ above\ values,\ \ we\ have\quad \frac{\boldsymbol{a}}{\boldsymbol{2\sqrt{6}}} < \frac{\boldsymbol{a}}{\left(\boldsymbol{8\pi\sqrt{2}}\right)^{1/3}} < \frac{\boldsymbol{a}}{\boldsymbol{2}}\sqrt{\frac{\boldsymbol{3}}{\boldsymbol{2}}}$$

$$\therefore\ \boldsymbol{R_i < R_m < R_o}$$

Mean radius (R_m) indicates that a regular tetrahedron with edge length a is equivalent to a sphere having a radius equal to $R_m = {a}/{\left(8\pi\sqrt{2}\right)^{1/3}}$ i.e. a regular tetrahedron can be replaced by a sphere having a radius equal to the mean radius (R_m) of given regular tetrahedron for the same volume.

4. Surface area (A_s)

$$\boldsymbol{A_s} = \frac{1}{4}nn_f a^2 cot\frac{\pi}{n} = \frac{1}{4}(3\times 4)a^2 cot\frac{\pi}{3} = \ 3a^2\frac{1}{\sqrt{3}} = \sqrt{3}\boldsymbol{a^2} \approx \boldsymbol{1.732050808a^2}$$

5. Volume (V)

$$V = \frac{1}{24}nn_f a^3 cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}} = \frac{1}{24}(3\times 4)a^3 cot\frac{\pi}{3}\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - cot^2\frac{\pi}{3}}$$

$$\boldsymbol{V} = \frac{a^3}{2}\times\frac{1}{\sqrt{3}}\sqrt{\frac{1}{2}-\frac{1}{3}} = \frac{a^3}{2\sqrt{3}}\sqrt{\frac{1}{6}} = \frac{\boldsymbol{a^3}}{\boldsymbol{6\sqrt{2}}} \approx \boldsymbol{0.11785113a^3}$$

Analysis of Regular Hexahedron (Cube): (Verification of HCR's Formula)

We are much familiar with the geometry of a cube & the results obtained relevant to a cube can be very easily verified. A **cube** is also a type of regular polyhedron called **regular hexahedron having six congruent faces each as a square**. It can be easily constructed by joining six congruent right pyramids with base as a square & edge angle (α) angle (i.e. angle between lateral edges of elementary right pyramid). Now, let there be a regular hexahedron (cube) having each edge of length a. We can easily find out all the parameters of a regular tetrahedron (cube) by using HCR's formula to calculate edge angle α as follows

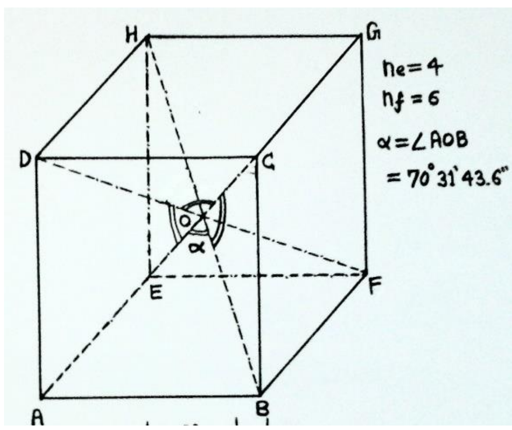

Figure 3: Regular Hexahedron (Cube) having six congruent square faces $(\boldsymbol{n = 4}\ \&\ \boldsymbol{n_f = 6})$

$$\alpha = 2\tan^{-1}\left(sec\frac{\pi}{n}\sqrt{sin\left\{\frac{2\pi}{nn_f}\right\}sin\left\{\frac{2\pi(n_f-1)}{nn_f}\right\}}\right) \quad \forall\, \alpha \le \frac{2\pi}{n}$$

In this case of a regular hexahedron (cube), we have

$n = no.of\ edges\ in\ one\ face = 4\ \ \&\ \ n_f = no.of\ faces = 6$

Now, setting both these integer values in HCR's Formula, we get

$$\alpha = 2\tan^{-1}\left(sec\frac{\pi}{4}\sqrt{sin\left\{\frac{2\pi}{4\times 6}\right\}sin\left\{\frac{2\pi(6-1)}{4\times 6}\right\}}\right) = 2\tan^{-1}\left(sec\frac{\pi}{4}\sqrt{sin\left\{\frac{\pi}{12}\right\}sin\left\{\frac{5\pi}{12}\right\}}\right)$$

$$= 2\tan^{-1}\left(\sqrt{2}\sqrt{sin15^o sin75^o}\right) = 2\tan^{-1}\left(\sqrt{2sin75^o sin15^o}\right)$$

$$= 2\tan^{-1}\left(\sqrt{cos(75^o-15^o)-cos(75^o+15^o)}\right) \qquad (since,\ \ 2sinAsinB = \cos(A-B)-\cos(A+B))$$

$$= 2\tan^{-1}\left(\sqrt{cos60^o}\right) = 2\tan^{-1}\left(\sqrt{\frac{1}{2}}\right) = \mathbf{2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 2\cot^{-1}(\sqrt{2}) \approx 70.52877937^o}$$

The above value of edge angle (α) can be experimentally verified by measuring it with the accuracy.

Now, by setting $\alpha = 2\cot^{-1}(\sqrt{2}), n = 4,\ n_f = 6\ \&\ edge\ length = a,$ all the parameters of a regular hexahedron (cube) are calculated as follows

1. Inner (inscribed) radius (R_i)

$$\boldsymbol{R_i} = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2}-cot^2\frac{\pi}{n}} = \frac{a}{2}\sqrt{\left(\sqrt{2}\right)^2-cot^2\frac{\pi}{4}} = \frac{a}{2}\sqrt{2-(1)^2} = \frac{a}{2}\sqrt{1} = \boldsymbol{\frac{a}{2} = 0.5a}$$

Hence, the **volume ($V_{inscribed}$) of the largest sphere inscribed/trapped in a regular hexahedron (cube)**

$$\Rightarrow \boldsymbol{V_{inscribed}} = \frac{4}{3}\pi(R_i)^3 = \frac{4}{3}\pi\left(\frac{a}{2}\right)^3 = \boldsymbol{\frac{\pi a^3}{6} \approx 0.523598775a^3}$$

2. Outer (circumscribed) radius (R_o)

$$\boldsymbol{R_o} = \frac{a}{2}cosec\frac{\alpha}{2} = \frac{a}{2}\sqrt{1+\left(cot\frac{\alpha}{2}\right)^2} = \frac{a}{2}\sqrt{1+\left(\sqrt{2}\right)^2} = \boldsymbol{\frac{a\sqrt{3}}{2} \approx 0.866025403a}$$

Hence, the **volume ($V_{circumscribed}$) of the smallest sphere circumscribing a regular tetrahedron (cube)**

$$\Rightarrow \boldsymbol{V_{circumscribed}} = \frac{4}{3}\pi(R_o)^3 = \frac{4}{3}\pi\left(\frac{a\sqrt{3}}{2}\right)^3 = \boldsymbol{\frac{\sqrt{3}\pi a^3}{2} \approx 2.720699046a^3}$$

3. Mean radius (R_m)

$$R_m = a\left(\frac{nn_f}{32\pi}cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2}-cot^2\frac{\pi}{n}}\right)^{\frac{1}{3}} = a\left(\frac{4\times 6}{32\pi}cot\frac{\pi}{4}\sqrt{\left(\sqrt{2}\right)^2-cot^2\frac{\pi}{4}}\right)^{\frac{1}{3}}$$

$$= a\left(\frac{3}{4\pi}\sqrt{2-(1)^2}\right)^{\frac{1}{3}} = \boldsymbol{a\left(\frac{3}{4\pi}\right)^{\frac{1}{3}} \approx 0.62035049a}$$

$$from\ above\ values,\quad we\ have\quad \frac{\boldsymbol{a}}{\boldsymbol{2}} < \boldsymbol{a}\left(\frac{\boldsymbol{3}}{\boldsymbol{4\pi}}\right)^{\frac{1}{3}} < \frac{\boldsymbol{a}\sqrt{\boldsymbol{3}}}{\boldsymbol{2}}$$

$$\therefore\ \boldsymbol{R_i < R_m < R_o}$$

Mean radius (R_m) indicates that a regular hexahedron (cube) with edge length a is equivalent to a sphere having a radius equal to the mean radius (R_m) i.e. a regular hexahedron can be replaced by a sphere having a radius equal to the mean radius (R_m) of that regular hexahedron (cube) for the same volume.

4. Surface area (A_s)

$$\boldsymbol{A_s} = \frac{1}{4}nn_f a^2 cot\frac{\pi}{n} = \frac{1}{4}(4\times 6)a^2 cot\frac{\pi}{4} = \boldsymbol{6a^2 = 6\times(edge\ length\ of\ cube)^2}$$

5. Volume (V)

$$\boldsymbol{V} = \frac{1}{24}nn_f a^3 cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}} = \frac{1}{24}(4\times 6)a^3 cot\frac{\pi}{4}\sqrt{\left(\sqrt{2}\right)^2 - cot^2\frac{\pi}{4}}$$

$$= a^3\sqrt{2-1} = \boldsymbol{a^3} = (\boldsymbol{edge\ length\ of\ cube})^{\boldsymbol{3}}$$

All the above results of a cube (i.e. regular hexahedron) are well known & are correct. Hence, HCR's Formula for regular polyhedron is verified on the basis of all the results obtained above

Analysis of Regular Octahedron (with eight faces each as an equilateral triangle)

Let there be a **regular octahedron** having **eight congruent faces each as an equilateral triangle** & edge length a. It can be easily constructed by joining eight congruent right pyramids with base as an equilateral triangle having each side equal to edge length a & edge angle (α) (i.e. angle between lateral edges of elementary right pyramid). We can easily find out all the parameters of a regular octahedron by using HCR's formula to calculate edge angle α as follows

$$\alpha = 2\tan^{-1}\left(sec\frac{\pi}{n}\sqrt{sin\left\{\frac{2\pi}{nn_f}\right\}sin\left\{\frac{2\pi(n_f-1)}{nn_f}\right\}}\right)\quad \forall\ \alpha \le \frac{2\pi}{n}$$

In this case of a regular octahedron, we have

$$n = no.of\ edges\ in\ one\ face = 3\ \ \&\ \ n_f = no.of\ faces = 8$$

Now, setting both these integer values in HCR's Formula, we get

$$\alpha = 2\tan^{-1}\left(sec\frac{\pi}{3}\sqrt{sin\left\{\frac{2\pi}{3\times 8}\right\}sin\left\{\frac{2\pi(8-1)}{3\times 8}\right\}}\right) = 2\tan^{-1}\left(2\sqrt{sin\left\{\frac{\pi}{12}\right\}sin\left\{\frac{7\pi}{12}\right\}}\right)$$

$$= 2\tan^{-1}\left(2\sqrt{sin15^o sin105^o}\right) = 2\tan^{-1}\left(\sqrt{2}\sqrt{2sin105^o sin15^o}\right)$$

$$= 2\tan^{-1}\left(\sqrt{2}\sqrt{\cos(105^o-15^o)-\cos(105^o+15^o)}\right) = 2\tan^{-1}\left(\sqrt{2}\sqrt{\cos 90^o - \cos120^o}\right)$$

$$= 2\tan^{-1}\left(\sqrt{2\left(0-\left(-\frac{1}{2}\right)\right)}\right) = 2\tan^{-1}\left(\sqrt{1}\right) = 2\tan^{-1}(1) = 90^o$$

$$\Rightarrow \boldsymbol{\alpha = 90^o} \Rightarrow \boldsymbol{tan\frac{\alpha}{2} = cot\frac{\alpha}{2} = 1}$$

Now, by substituting the value of α, $n = 3$, $n_f = 8$ & $edge\ length = a$, all the parameters of a regular octahedron are calculated as follows

1. Inner (inscribed) radius (R_i)

$$\boldsymbol{R_i} = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}} = \frac{a}{2}\sqrt{(1)^2 - cot^2\frac{\pi}{3}} = \frac{a}{2}\sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{a}{2}\sqrt{\frac{3-1}{3}} = \frac{a}{2}\sqrt{\frac{2}{3}} = \boldsymbol{\frac{a}{\sqrt{6}} \approx 0.40824829a}$$

Hence, the **volume ($V_{inscribed}$) of the largest sphere inscribed/trapped in a regular octahedron**

$$\Rightarrow \boldsymbol{V_{inscribed}} = \frac{4}{3}\pi(R_i)^3 = \frac{4}{3}\pi\left(\frac{a}{\sqrt{6}}\right)^3 = \boldsymbol{\frac{\pi a^3}{9}\sqrt{\frac{2}{3}} \approx 0.285011073a^3}$$

2. Outer (circumscribed) radius (R_o)

$$\boldsymbol{R_o} = \frac{a}{2}cosec\frac{\alpha}{2} = \frac{a}{2}cosec45^o = \frac{a}{2} \times \sqrt{2} = \boldsymbol{\frac{a}{\sqrt{2}} \approx 0.707106781a}$$

Hence, the **volume ($V_{circumscribed}$) of the smallest sphere circumscribing a regular octahedron**

$$\Rightarrow \boldsymbol{V_{circumscribed}} = \frac{4}{3}\pi(R_o)^3 = \frac{4}{3}\pi\left(\frac{a}{\sqrt{2}}\right)^3 = \boldsymbol{\frac{\pi\sqrt{2}}{3}a^3 \approx 1.480960979a^3}$$

3. Mean radius (R_m)

$$\boldsymbol{R_m} = a\left(\frac{nn_f}{32\pi}cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}\right)^{\frac{1}{3}} = a\left(\frac{3 \times 8}{32\pi}cot\frac{\pi}{3}\sqrt{(1)^2 - cot^2\frac{\pi}{3}}\right)^{\frac{1}{3}}$$

$$= a\left(\frac{3}{4\pi}\left(\frac{1}{\sqrt{3}}\right)\sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2}\right)^{\frac{1}{3}} = a\left(\frac{3}{4\pi}\left(\frac{1}{\sqrt{3}}\right)\sqrt{\frac{2}{3}}\right)^{\frac{1}{3}} = a\left(\frac{1}{2\pi\sqrt{2}}\right)^{\frac{1}{3}} = \boldsymbol{\frac{a}{(2\pi\sqrt{2})^{\frac{1}{3}}} \approx 0.482801241a}$$

$$from\ above\ values,\ \ we\ have\ \boldsymbol{R_i < R_m < R_o}$$

Mean radius (R_m) indicates that a regular octahedron with edge length a is equivalent to a sphere having a radius equal to $R_m = 0.482801241a$ i.e. a regular octahedron can be replaced by a sphere having a radius equal to the mean radius (R_m) of that regular octahedron for the same volume.

4. Surface area (A_s)

$$\boldsymbol{A_s} = \frac{1}{4}nn_f a^2 cot\frac{\pi}{n} = \frac{1}{4}(3 \times 8)a^2 cot\frac{\pi}{3} = 6a^2\left(\frac{1}{\sqrt{3}}\right) = \boldsymbol{2\sqrt{3}a^2 \approx 3.464101615a^2}$$

5. Volume (V)

$$\boldsymbol{V}=\frac{1}{24}nn_{f}a^{3}cot\frac{\pi}{n}\sqrt{cot^{2}\frac{\alpha}{2}-cot^{2}\frac{\pi}{n}}=\frac{1}{24}(3\times 8)a^{3}cot\frac{\pi}{3}\sqrt{(1)^{2}-cot^{2}\frac{\pi}{3}}$$

$$=a^{3}\left(\frac{1}{\sqrt{3}}\right)\sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^{2}}=\left(\frac{a^{3}}{\sqrt{3}}\right)\sqrt{\frac{2}{3}}=\frac{\sqrt{\mathbf{2}}}{\mathbf{3}}\boldsymbol{a}^{\mathbf{3}}\approx \mathbf{0.47140452}\boldsymbol{a}^{\mathbf{3}}$$

Analysis of Regular Dodecahedron with congruent regular pentagonal faces:

Let there be a regular dodecahedron having twelve congruent faces each as a regular pentagon & edge length a. It can be easily constructed by joining twelve congruent right pyramids with base as a regular pentagon having each side equal to edge length a & edge angle (α) (i.e. angle between lateral edges of elementary right pyramid). We can easily find out all the parameters of a regular dodecahedron by using HCR's formula to calculate edge angle α as follows

$$\alpha=2\tan^{-1}\left(sec\frac{\pi}{n}\sqrt{sin\left\{\frac{2\pi}{nn_{f}}\right\}sin\left\{\frac{2\pi(n_{f}-1)}{nn_{f}}\right\}}\right)\quad \forall\, \alpha\leq\frac{2\pi}{n}$$

In this case of a regular dodecahedron, we have

$$n=no.of\ edges\ in\ one\ face=5\ \ \&\ n_{f}=no.of\ faces=12$$

Now, setting both these integer values in HCR's Formula, we get

$$\alpha=2\tan^{-1}\left(sec\frac{\pi}{5}\sqrt{sin\left\{\frac{2\pi}{5\times 12}\right\}sin\left\{\frac{2\pi(12-1)}{5\times 12}\right\}}\right)=2\tan^{-1}\left(sec\frac{\pi}{5}\sqrt{sin\left\{\frac{\pi}{30}\right\}sin\left\{\frac{11\pi}{30}\right\}}\right)$$

$$=2\tan^{-1}\left(sec36^{o}\sqrt{sin6^{o}sin66^{o}}\right)=2\tan^{-1}\left(\frac{1}{\sqrt{2}}sec36^{o}\sqrt{2sin66^{o}sin6^{o}}\right)$$

$$=2\tan^{-1}\left(\frac{1}{\sqrt{2}}sec36^{o}\sqrt{\cos(66^{o}-6^{o})-\cos(66^{o}+6^{o})}\right)=2\tan^{-1}\left(\frac{1}{\sqrt{2}}\left(\frac{1}{cos36^{o}}\right)\sqrt{\cos 60^{o}-\cos72^{o}}\right)$$

$$=2\tan^{-1}\left(\sqrt{\frac{\frac{1}{2}-\sin18^{o}}{2cos^{2}36^{o}}}\right)=2\tan^{-1}\left(\sqrt{\frac{\frac{1}{2}-\left(\frac{\sqrt{5}-1}{4}\right)}{2\left(\frac{\sqrt{5}+1}{4}\right)^{2}}}\right)\quad (substituting\ the\ values\ of\ sin18^{o}\ \&\ cos36^{o})$$

$$=2\tan^{-1}\left(\sqrt{\frac{\left(\frac{2-\sqrt{5}+1}{4}\right)}{2\left(\frac{5+2\sqrt{5}+1}{16}\right)}}\right)=2\tan^{-1}\left(\sqrt{\frac{(3-\sqrt{5})}{4\left(\frac{3+\sqrt{5}}{4}\right)}}\right)=2\tan^{-1}\left(\sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}}\right)$$

$$=2\tan^{-1}\left(\sqrt{\frac{(3-\sqrt{5})^{2}}{(3+\sqrt{5})(3-\sqrt{5})}}\right)=2\tan^{-1}\left(\sqrt{\frac{(3-\sqrt{5})^{2}}{4}}\right)=2\tan^{-1}\left(\frac{3-\sqrt{5}}{2}\right)$$

$$\Rightarrow\ \boldsymbol{\alpha}=\mathbf{2}\tan^{-1}\left(\frac{\mathbf{3}-\sqrt{\mathbf{5}}}{\mathbf{2}}\right)=\mathbf{2}\cot^{-1}\left(\frac{\mathbf{3}+\sqrt{\mathbf{5}}}{\mathbf{2}}\right)\approx \mathbf{41.8103149}^{\boldsymbol{o}}$$

Now, by substituting the value of α, $n = 5$, $n_f = 12$ & $edge\ length = a$, all the parameters of a regular dodecahedron are calculated as follows

1. Inner (inscribed) radius (R_i)

$$\boldsymbol{R_i} = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}} = \frac{a}{2}\sqrt{\left(\frac{3+\sqrt{5}}{2}\right)^2 - cot^2\frac{\pi}{5}} = \frac{a}{2}\sqrt{\frac{9+5+6\sqrt{5}}{4} - \left(\frac{cos36^o}{sin36^o}\right)^2}$$

$$= \frac{a}{2}\sqrt{\frac{7+3\sqrt{5}}{2} - \left(\frac{\left(\frac{\sqrt{5}+1}{4}\right)}{\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)}\right)^2} = \frac{a}{2}\sqrt{\frac{7+3\sqrt{5}}{5-\sqrt{5}}} = \frac{a}{2}\sqrt{\frac{14+6\sqrt{5}}{10-2\sqrt{5}}} = \frac{a}{2}\sqrt{\frac{\left(3+\sqrt{5}\right)^2}{10-2\sqrt{5}}}$$

$$= \frac{\boldsymbol{(3+\sqrt{5})a}}{\boldsymbol{2\sqrt{10-2\sqrt{5}}}} = \frac{\boldsymbol{(3+\sqrt{5})a}}{\boldsymbol{8sin36^o}} \approx \boldsymbol{1.113516364a}$$

Hence, the **volume** ($\boldsymbol{V_{inscribed}}$) **of the largest sphere inscribed/trapped in a regular dodecahedron**

$$\Rightarrow \boldsymbol{V_{inscribed}} = \frac{4}{3}\pi(R_i)^3 = \frac{4}{3}\pi\left(\frac{(3+\sqrt{5})a}{2sin36^o}\right)^3 \approx \boldsymbol{5.783335944a^3}$$

2. Outer (circumscribed) radius (R_o)

$$\boldsymbol{R_o} = \frac{a}{2}cosec\frac{\alpha}{2} = \frac{a}{2}\sqrt{1+\left(cot\frac{\alpha}{2}\right)^2} = \frac{a}{2}\sqrt{1+\left(\frac{3+\sqrt{5}}{2}\right)^2} = \frac{a}{2}\sqrt{\frac{9+3\sqrt{5}}{2}} = \frac{a}{2}\sqrt{\frac{3\left(6+2\sqrt{5}\right)}{4}}$$

$$= \frac{a\sqrt{3}}{4}\sqrt{\left(\sqrt{5}+1\right)^2} = \frac{\boldsymbol{\sqrt{3}(\sqrt{5}+1)a}}{\boldsymbol{4}} \approx \boldsymbol{1.401258538a}$$

Hence, the **volume** ($\boldsymbol{V_{circumscribed}}$) **of the smallest sphere circumscribing a regular dodecahedron**

$$\Rightarrow \boldsymbol{V_{circumscribed}} = \frac{4}{3}\pi(R_o)^3 = \frac{4}{3}\pi\left(\frac{\sqrt{3}(\sqrt{5}+1)a}{4}\right)^3 \approx \boldsymbol{11.5250661a^3}$$

3. Mean radius (R_m)

$$R_m = a\left(\frac{nn_f}{32\pi}cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}\right)^{\frac{1}{3}} = a\left(\frac{5\times 12}{32\pi}cot\frac{\pi}{5}\sqrt{\left(\frac{3+\sqrt{5}}{2}\right)^2 - cot^2\frac{\pi}{5}}\right)^{\frac{1}{3}}$$

$$= a\left(\frac{15}{8\pi}\left(\frac{cos36^o}{sin36^o}\right)\sqrt{\frac{9+5+6\sqrt{5}}{4} - \left(\frac{cos36^o}{sin36^o}\right)^2}\right)^{\frac{1}{3}}$$

$$= a\left(\frac{15}{8\pi}\left(\frac{\left(\frac{\sqrt{5}+1}{4}\right)}{\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)}\right)\sqrt{\frac{7+3\sqrt{5}}{2} - \left(\frac{\left(\frac{\sqrt{5}+1}{4}\right)}{\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)}\right)^2}\right)^{=\frac{1}{3}} = a\left(\frac{15}{8\pi}\left(\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}\right)\sqrt{\frac{7+3\sqrt{5}}{5-\sqrt{5}}}\right)^{\frac{1}{3}}$$

$$= a\left(\frac{15}{8\pi}\left(\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}\right)\sqrt{\frac{14+6\sqrt{5}}{10-2\sqrt{5}}}\right)^{\frac{1}{3}} = a\left(\frac{15}{8\pi}\left(\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}\right)\frac{\sqrt{\left(3+\sqrt{5}\right)^2}}{\sqrt{10-2\sqrt{5}}}\right)^{\frac{1}{3}}$$

$$= a\left(\frac{15}{8\pi}\left(\frac{\left(\sqrt{5}+1\right)\left(3+\sqrt{5}\right)}{10-2\sqrt{5}}\right)\right)^{\frac{1}{3}} = a\left(\frac{15}{4\pi}\left(\frac{2+\sqrt{5}}{5-\sqrt{5}}\right)\right)^{\frac{1}{3}} = a\left(\frac{15}{4\pi}\frac{\left(2+\sqrt{5}\right)\left(5+\sqrt{5}\right)}{\left(5-\sqrt{5}\right)\left(5+\sqrt{5}\right)}\right)^{\frac{1}{3}}$$

$$= a\left(\frac{15}{4\pi}\frac{\left(15+7\sqrt{5}\right)}{(25-5)}\right)^{\frac{1}{3}} = a\left(\frac{15\left(15+7\sqrt{5}\right)}{4\pi\times 20}\right)^{\frac{1}{3}} = \boldsymbol{a\left(\frac{3\left(15+7\sqrt{5}\right)}{16\pi}\right)^{\frac{1}{3}}} \approx \boldsymbol{1.223035283a}$$

$$from\ above\ values,\ \ we\ have\ \boldsymbol{R_i < R_m < R_o}$$

Mean radius (R_m) indicates that a regular dodecahedron with edge length a is equivalent to a sphere having a radius equal to $R_m = 1.223035283a$ i.e. a regular dodecahedron can be replaced by a sphere having a radius equal to the mean radius (R_m) of that regular dodecahedron for the same volume.

4. Surface area (A_s)

$$\boldsymbol{A_s} = \frac{1}{4}nn_f a^2 cot\frac{\pi}{n} = \frac{1}{4}(5\times 12)a^2 cot\frac{\pi}{5} = 15a^2\left(\frac{cos36^o}{sin36^o}\right) = 15a^2\left(\frac{\left(\frac{\sqrt{5}+1}{4}\right)}{\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)}\right)$$

$$= 15a^2\left(\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}\right) = \boldsymbol{15\left(\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}\right)a^2} \approx \boldsymbol{20.64572881a^2}$$

5. Volume (V)

$$\boldsymbol{V} = \frac{1}{24}nn_f a^3 cot\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}} = \frac{1}{24}(5\times 12)a^3 cot\frac{\pi}{5}\sqrt{\left(\frac{3+\sqrt{5}}{2}\right)^2 - cot^2\frac{\pi}{5}}$$

$$= \frac{5}{2}a^3\left(\frac{cos36^o}{sin36^o}\right)\sqrt{\frac{9+5+6\sqrt{5}}{4} - \left(\frac{cos36^o}{sin36^o}\right)^2} = \boldsymbol{\frac{\left(15+7\sqrt{5}\right)}{4}a^3} \approx \boldsymbol{7.663118961a^3}$$

There is only one regular polyhedron having all its faces as congruent regular pentagons called regular dodecahedron which has twelve faces as congruent regular pentagons & there is no regular polyhedron having all its faces as congruent regular hexagons in the Universe.

Analysis of Regular Icosahedron (with twenty faces each as an equilateral triangle):

Let there be a **regular icosahedron** having **twenty congruent faces each as an equilateral triangle** & edge length a. It can be easily constructed by joining twenty congruent right pyramids with base as an equilateral triangle having each side equal to edge length a & edge angle (α) (i.e. angle between lateral edges of elementary right pyramid). We can easily find out all the parameters of a regular icosahedron by using HCR's formula to calculate edge angle α as follows

$$\alpha = 2\tan^{-1}\left(sec\frac{\pi}{n}\sqrt{sin\left\{\frac{2\pi}{nn_f}\right\}sin\left\{\frac{2\pi(n_f-1)}{nn_f}\right\}}\right) \quad \forall\, \alpha \le \frac{2\pi}{n}$$

In this case of a regular icosahedron, we have

$$n = no.of\ edges\ in\ one\ face = 3\ \ \&\ \ n_f = no.of\ faces = 20$$

Now, setting both these integer values in HCR's Formula, we get

$$\alpha = 2\tan^{-1}\left(sec\frac{\pi}{3}\sqrt{sin\left\{\frac{2\pi}{3\times 20}\right\}sin\left\{\frac{2\pi(20-1)}{3\times 20}\right\}}\right) = 2\tan^{-1}\left(2\sqrt{sin\left\{\frac{\pi}{30}\right\}sin\left\{\frac{19\pi}{30}\right\}}\right)$$

$$= 2\tan^{-1}\left(2\sqrt{sin6^o sin114^o}\right) = 2\tan^{-1}\left(\sqrt{2}\sqrt{2sin114^o sin6^o}\right)$$

$$= 2\tan^{-1}\left(\sqrt{2}\sqrt{\cos(114^o-6^o)-\cos(114^o+6^o)}\right) = 2\tan^{-1}\left(\sqrt{2}\sqrt{\cos(90^o+18^o)-\cos120^o}\right)$$

$$= 2\tan^{-1}\left(\sqrt{2}\sqrt{-\sin18^o+\cos60^o}\right) = 2\tan^{-1}\left(\sqrt{2}\sqrt{\frac{1}{2}-\frac{\sqrt{5}-1}{4}}\right) = 2\tan^{-1}\left(\sqrt{2}\sqrt{\frac{2-\sqrt{5}+1}{4}}\right)$$

$$or\ \ \boldsymbol{\alpha = 2\tan^{-1}\left(\sqrt{\frac{3-\sqrt{5}}{2}}\right) = 2\cot^{-1}\left(\sqrt{\frac{3+\sqrt{5}}{2}}\right) \approx 63.43494882^o}$$

Now, by substituting the value of α, $n = 3$, $n_f = 20$ & $edge\ length = a$, all the parameters of a regular icosahedron are calculated as follows

1. Inner (inscribed) radius (R_i)

$$\boldsymbol{R_i} = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2}-cot^2\frac{\pi}{n}} = \frac{a}{2}\sqrt{\left(\sqrt{\frac{3+\sqrt{5}}{2}}\right)^2 - cot^2\frac{\pi}{3}} = \frac{a}{2}\sqrt{\frac{3+\sqrt{5}}{2}-\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{a}{2}\sqrt{\frac{9+3\sqrt{5}-2}{6}}$$

$$= \frac{a}{2}\sqrt{\frac{7+3\sqrt{5}}{6}} = \frac{a}{2}\sqrt{\frac{14+6\sqrt{5}}{12}} = \frac{a}{4}\sqrt{\frac{\left(3+\sqrt{5}\right)^2}{3}} = \boldsymbol{\left(\frac{3+\sqrt{5}}{4\sqrt{3}}\right)a \approx 0.755761314a}$$

Hence, the **volume ($V_{inscribed}$) of the largest sphere inscribed/trapped in a regular icosahedron**

$$\Rightarrow \boldsymbol{V_{inscribed}} = \frac{4}{3}\pi(R_i)^3 = \boldsymbol{\frac{4}{3}\pi\left(\left(\frac{3+\sqrt{5}}{4\sqrt{3}}\right)a\right)^3 \approx 1.808183832a^3}$$

2. Outer (circumscribed) radius (R_o)

$$\boldsymbol{R_o} = \frac{a}{2}cosec\frac{\alpha}{2} = \frac{a}{2}\sqrt{1+cot^2\frac{\alpha}{2}} = \frac{a}{2}\sqrt{1+\left(\sqrt{\frac{3+\sqrt{5}}{2}}\right)^2} = \frac{a}{2}\sqrt{\frac{2+3+\sqrt{5}}{2}}$$

$$=\frac{a}{2}\sqrt{\frac{5+\sqrt{5}}{2}}=\frac{\boldsymbol{a\sqrt{10+2\sqrt{5}}}}{\boldsymbol{4}}=\boldsymbol{a\cos 18^o\approx 0.951056516a}$$

Hence, the **volume** $(V_{circumscribed})$ **of the smallest sphere circumscribing a regular icosahedron**

$$\Rightarrow \boldsymbol{V_{circumscribed}}=\frac{4}{3}\pi(R_o)^3=\frac{\boldsymbol{4}}{\boldsymbol{3}}\boldsymbol{\pi}\left(\frac{\boldsymbol{a\sqrt{10+2\sqrt{5}}}}{\boldsymbol{4}}\right)^{\boldsymbol{3}}\approx \boldsymbol{3.603359442a^3}$$

3. Mean radius (R_m)

$$R_m=a\left(\frac{nn_f}{32\pi}\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}\right)^{\frac{1}{3}}=a\left(\frac{3\times 20}{32\pi}\cot\frac{\pi}{3}\sqrt{\left(\sqrt{\frac{3+\sqrt{5}}{2}}\right)^2-\cot^2\frac{\pi}{3}}\right)^{\frac{1}{3}}$$

$$=a\left(\frac{15}{8\pi}\left(\frac{1}{\sqrt{3}}\right)\sqrt{\frac{3+\sqrt{5}}{2}-\frac{1}{3}}\right)^{\frac{1}{3}}=a\left(\frac{5\sqrt{3}}{8\pi}\sqrt{\frac{9+3\sqrt{5}-2}{6}}\right)^{\frac{1}{3}}$$

$$=a\left(\frac{5}{8\pi}\sqrt{\frac{7+3\sqrt{5}}{2}}\right)^{\frac{1}{3}}=a\left(\frac{5}{8\pi}\sqrt{\frac{(3+\sqrt{5})^2}{4}}\right)^{\frac{1}{3}}=\boldsymbol{a\left(\frac{5(3+\sqrt{5})}{16\pi}\right)^{\frac{1}{3}}}\approx\boldsymbol{0.804578596a}$$

$$from\ above\ values,\ \ we\ have\ \ \boldsymbol{R_i<R_m<R_o}$$

Mean radius (R_m) indicates that a regular icosahedron with edge length a is equivalent to a sphere having a radius equal to $R_m=0.804578596$ i.e. a regular icosahedron can be replaced by a sphere having a radius equal to the mean radius (R_m) of that regular icosahedron for the same volume.

4. Surface area (A_s)

$$\boldsymbol{A_s}=\frac{1}{4}nn_f a^2\cot\frac{\pi}{n}=\frac{1}{4}(3\times 20)a^2\cot\frac{\pi}{3}=15a^2\left(\frac{1}{\sqrt{3}}\right)=\boldsymbol{5\sqrt{3}a^2\approx 8.660254038a^2}$$

5. Volume (V)

$$\boldsymbol{V}=\frac{1}{24}nn_f a^3\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}=\frac{1}{24}(3\times 20)a^3\cot\frac{\pi}{3}\sqrt{\left(\sqrt{\frac{3+\sqrt{5}}{2}}\right)^2-\cot^2\frac{\pi}{3}}$$

$$=\frac{5}{2}a^3\left(\frac{1}{\sqrt{3}}\right)\sqrt{\frac{3+\sqrt{5}}{2}-\frac{1}{3}}=\left(\frac{5a^3}{2\sqrt{3}}\right)\sqrt{\frac{7+3\sqrt{5}}{6}}=\left(\frac{5a^3}{6}\right)\sqrt{\frac{14+6\sqrt{5}}{4}}$$

$$=\left(\frac{5a^3}{6}\right)\sqrt{\frac{(3+\sqrt{5})^2}{4}}=\boldsymbol{\frac{5}{12}(3+\sqrt{5})a^3\approx 2.181694991a^3}$$

Conclusion: On the basis of above results, it is concluded that HCR's Formula can be applied on **any of five existing regular n-polyhedrons (i.e. platonic solids) The most important parameter of any regular n-polyhedron is the edge angle (α) of its congruent elementary right pyramids which is independent of the geometrical dimensions of regular polyhedron & this particular parameter is easily calculated simply by**

counting the no. of edges in one face (n) & the no. of faces (n_f). It also concludes that there are only three governing parameters of any regular polyhedron as the no. of edges in one face (n), the no. of faces (n_f) & the edge length (a) if these parameters are known we can calculate all the important parameters of any regular polyhedron.

Applications: This formula is very useful to calculate all the important parameters of any **of five regular n-polyhedrons, having all its faces as the congruent regular n-polygons**, such as **inner radius, outer radius, mean radius, surface area & volume. It can be used in analysis, designing & modelling of regular n-polyhedrons.**

Remember: This formula is applicable on any of five regular polyhedrons (i.e. platonic solids) which has

1. **Its convex surface &**
2. **All its faces as congruent regular n-polygons**

Thus, this formula is applicable on any of the five platonic solids i.e. regular tetrahedron, cube, regular octahedron, regular icosahedron & regular dodecahedron. It is not applicable on the polyhedrons having dissimilar faces or concave surfaces i.e. irregular polyhedrons because a regular polyhedron always has convex surface. Regular stars are not the regular polyhedrons (platonic solids) because these have concave surfaces. In these cases, this formula is not applicable.

Note: Above articles had been developed *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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